TIMES Micro – Elastic Demand Functions

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Contents

1 Introduction

This document describes a draft design for generalizing the available TIMES Demand Function formulations, focusing on the demand elasticities. Until TIMES v4.0, only the linearized own-price elasticity formulation was available in the common code. The corresponding non-linear formulation, which was available in MARKAL (see Loulou & al. 2004), could well be introduced into TIMES as well, as suggested in the draft design.

When substitution possibilities are to be modeled, demand functions involving Constant Elasticity of Substitution (CES) aggregates are proposed to be made available under the non-linear option. The formulation for including the CES aggregates is based on the old sketches found in the MARKAL GAMS code (but not active in the code), apparently designed by Dr. Denise van Regemorter and implemented by Gary Goldstein. Just as under the own-price elasticity option, the calibration of the CES functions is based on the demand projections and the corresponding shadow prices from a Baseline TIMES run, with the substitution elasticity between the demands within each category given as an input, together with the own-price elasticity for the whole demand category.

Finally, a draft design for a simple linearization of the CES utility function formulation is proposed and presented.

All the simple generalizations drafted in this note have been implemented in TIMES v4.1.0. For now, the implementation should be still considered experimental, and therefore any feedback and comments from TIMES users concerning the formulation and implementation are welcome.

2 Non-Linear Demand Functions

2.1 Demand functions with own-price elasticities

In the TIMES partial equilibrium formulation, each energy service demand is assumed to have a constant own price elasticity function of the form:

$$\frac{DM_i}{DM_i^0} = \left(\frac{p_i}{p_i^0}\right)^E \tag{1}$$

Where $\{DM^0, P^0\}$ is a reference pair of demand and price values for that energy service over the forecast horizon, and E is the (negative) own price elasticity of that energy service demand, as chosen by the user (note that though not shown by the notation, this price elasticity may vary over time). The pair $\{DM^0, P^0\}$ is obtained by solving TIMES for a Baseline scenario. More precisely, DM^0 is the demand projection estimated by the user in the reference case based upon explicitly defined relationships to economic and demographic drivers, and P^0 is the shadow price of that energy service demand obtained by running the Baseline scenario of the TIMES model.

The objective function maximizes the total present value surplus of the consumers and producers, which is equivalent to minimizing the opposite number. The surplus is obtained by integrating the difference between the demand price functions and the supply cost functions from zero to the demand levels. The supply cost functions are integrated simply by taking the vector product $c^T \cdot X$ (in present value terms), and the demand price functions can also be easily integrated from the following representation:

$$p_i = p_i^0 \cdot \left(\frac{DM_i}{DM_i^0}\right)^{1/E} \tag{2}$$

The full objective function can thus be written as:

$$\begin{aligned} &Min\ \ c^T \cdot X - \sum_i \sum_t PVF(t) \cdot dc_i(t) \cdot DM_i(t)^{1+1/E_i} \\ &s.t.\ \sum_k VAR\_ACT_{k,i}(t) \geq DM_i(t) \quad i = 1,..,I; t = 1,..,T \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad B \cdot X \geq b \end{aligned} \tag{3}$$

Here, PVF is the present value factor and the constants dc_i can be calculated as

$$dc_i(t) = \frac{p_i^0(t)}{(1 + 1/E_i) \cdot \left(DM_i^0(t)\right)^{1/E_i}} \tag{4}$$

2.2 CES Utility Functions

Consider a utility function of the general CES form:

$$U = \left(\sum_i \alpha_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \qquad (5)$$

where:

- U is the total aggregate utility
- x_i is the demand for commodity i
- α_i is a share parameter (the sum of which over i needs not be equal to 1)
- σ is the elasticity of substitution ($0 < \sigma < \infty$)

The demand functions for x_i can be derived from the utility function in terms of prices, and can be given by the formulas:

$$x_i = \frac{\alpha_i m}{p_i^{\sigma}} \left(\sum_i \alpha_i \, p_i^{1-\sigma} \right)^{-1} = \frac{\alpha_i m}{p_u} \left(\frac{p_u}{p_i} \right)^{\sigma} \qquad (6)$$

where m is the income level, and p_u is the aggregate price, or unit cost, of the utility, can be given in terms of the individual prices p_i of the demands i:

$$p_u = \left(\sum_i \alpha_i \, p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \qquad (7)$$

However, because we also know that the unit cost of the utility is the total expenditure divided by the utility, we can calculate the unit cost at the Baseline solution:

$$p_u^* = \frac{\sum_i p_i^* x_i^*}{U^*} \qquad (8)$$

The share parameters α_i can be obtained from the expression for the expenditure shares s_i:

$$s_i = \alpha_i \cdot \left(\frac{p_u}{p_i} \right)^{\sigma-1} \qquad (9)$$

This relation also holds at the Baseline solution, where we can write:

$$\alpha_i = s_i \cdot \left(\frac{p_i^*}{p_u^*} \right)^{\sigma-1} = \frac{p_i^* x_i^*}{p_u^* U^*} \cdot \left(\frac{p_i^*}{p_u^*} \right)^{\sigma-1} = \frac{x_i^*}{U^*} \cdot \left(\frac{p_i^*}{p_u^*} \right)^{\sigma} \qquad (10)$$

One can see that we can choose any nominal value $U^* \geq 0$, and then the Base price p^*_u and the share parameters α_i can be derived for the utility function.

2.3 TIMES Objective Function with CES Utility Functions

The CES utility functions described in Section 2.2 can be used for combining subsets of the TIMES demands into aggregate CES demands. First, one should just choose a convenient measure for the aggregate utility in the Baseline solution $U^0 = U^*$, and then both the Base price $p^0_{u_k}$ for each U^0_k and the share parameters α_{ki} can be derived, to complete the TIMES Micro formulation supporting also aggregate demands in the form of CES utility functions.

For each aggregate demand k, let us define the values U^0_k to be equal to the sum of the scaled component demands DM_{ki} in the Baseline solution:

$$\begin{aligned} DM^0_{ki}(t) &= dag_{ki}(t) \cdot DM^0_i(t) \\ U^0_k(t) &= \sum_i dag_{ki}(t) \cdot DM^0_i(t) = \sum_i DM^0_{ki}(t) \end{aligned} \tag{11}$$

The full objective function can thus now be written as

$$\begin{aligned} & Min\ \ c^T \cdot X - \sum_i \sum_t PVF(t) \cdot dc_i(t) \cdot DM_i(t)^{1+1/E_i} - \sum_k \sum_t PVF(t) \cdot uc_k(t) \cdot U_k(t)^{1+1/E_k} \\ & \qquad s.t.\ \sum_k VAR\_ACT_{k,i}(t) \geq DM_i(t) \quad i = 1,..,I; t = 1,..,T \\ & \qquad B \cdot X \geq b \end{aligned} \tag{12}$$

Here, the constants dc_i and uc_k can be calculated as

$$\begin{aligned} dc_i(t) &= \frac{p^0_i(t)}{(1+1/E_i) \cdot \left(DM^0_i(t)\right)^{1/E_i}}, \quad \forall i : (\neg \exists k : dag_{ki}(t)) \\ uc_k(t) &= \frac{p^0_{u_k}(t)}{(1+1/E_k) \cdot \left(U^0_k(t)\right)^{1/E_k}}, \quad \forall k : (\exists i : dag_{ki}(t)) \end{aligned} \tag{13}$$

The utility functions $U_k(t)$ of course need not be explicitly represented by variables in TIMES, but can be written out in terms of the component demands:

$$U_k(t) = \left(\sum_j \alpha_{ki}(t)^{\frac{1}{\sigma}} DM_{kj}(t)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \tag{14}$$

The original demands DM_i corresponding to the component demands DM_{ki} of each aggregate demand U_k must also be excluded from the set of independently handled demands DM_i in the objective function (12), as indicated in (13).

The Baseline demands $DM^0_i(t)$, the own-price elasticities E_i / E_k, the substitution elasticities σ_k and the aggregation parameters $dag_{ki}(t)$ are all input parameters defined by the user. The Base prices $p^0_i(t)$ are obtained from the Baseline solution, and the $p^0_{u_k}$ prices and the α_i parameters are derived from the others.

3 Linearized Demand Elasticities

3.1 Demand functions with own-price elasticities

Since the terms $(DM_i(t))^{1+1/Ei}$ are non-linear expressions and thus not directly usable in an LP, a linear formulation has been developed for TIMES. The linearization proceeds as follows (adapted from the TIMES Documentation, Part I, see Loulou & al. 2016):

a) For each demand category *i*, and each time period *t*, the user selects a step size $ß_i(t)$ and number of steps m_i and n_i, defining the range $R(t)_i$, i.e. the distance between some values $DM_i(t)_{min}$ and $DM_i(t)_{max}$. The user estimates that the demand value $DM_i(t)$ will always remain within the range $DM^o{}_i(t) - m_i \cdot ß_i(t) \leq DM_i(t) \leq DM^o{}_i(t) + n_i \cdot ß_i(t)$ even after adjustment for price effects (for instance, the range could be equal to the reference demand $DM^o{}_i(t)$ plus or minus 50%).

b) Select a grid that divides each range into a number $m_i + n_i$ of equal width intervals. The number of steps should be chosen so that the step-wise constant approximation remains close to the exact value of the function.

c) For each demand segment $DM_i(t)$ define $m+n$ step-variables (one per grid interval), denoted $sm_{1,i}(t)$, $sm_{2,i}(t)$, …, $sm_{m,i}(t)$ and $sn_{1,i}(t)$, $sn_{2,i}(t)$, …, $sn_{n,i}(t)$. Each *s* variable is bounded below by 0 and above by $ß_i(t)$. One may now replace in the equations (3) and (12) each $DM_i(t)$ variable by the sum of the m-n step variables, and each non-linear term in the objective function by a weighted sum of the m-n step-variables, as follows:

$$DM_i(t) = DM_i^0(t) - \sum_{j=1}^{m} sm_{j,i}(t) + \sum_{j=1}^{n} sn_{j,i}(t) \qquad (15)$$

$$DM_i(t)^{1+1/E_i} \cong DM_i^0(t)^{1+1/E_i} - \sum_{j=1}^{m} AM_{j,i}(t) \bullet sm_{j,i}(t) + \sum_{j=1}^{n} AN_{j,i}(t) \bullet sn_{j,i}(t) \qquad (16)$$

Here, the marginal utility coefficients $AM_{j,i}$ and $AN_{j,i}$ for the variables $sm_{j,i}$ and $sn_{j,i}$ can be easily calculated from the demand price function at the mid point of each step *j*.

$$dc_i(t) \cdot AM_{j,i}(t) = dc_i(t) \cdot (1+1/E_i) \cdot DM_{ji}(t)^{1/E_i} = p_i^0(t)\left(\frac{DM_i^0 - (j-½) \times \beta_i}{DM_i^0}\right)^{1/E_i} = p_{ji}^-(t) \qquad (17)$$

$$dc_i(t) \cdot AN_{j,i}(t) = dc_i(t) \cdot (1+1/E_i) \cdot DM_{ji}(t)^{1/E_i} = p_i^0(t)\left(\frac{DM_i^0 + (j-½) \times \beta_i}{DM_i^0}\right)^{1/E_i} = p_{ji}^+(t) \qquad (18)$$

3.2 Demand functions with cross-price elasticities

As one can see from equation (12), we can linearize the aggregate demands $U_k(t)$ exactly in the same way as the independent demands $DM_i(t)$. However, for linearizing the complete demand functions involving CES utility functions, we additionally need to linearize the cross elasticities inside each CES aggregate. For that purpose, we can approximate each aggregate CES demand function by using an approach very similar to the formulations in Sections 2.3 and 3.1.

First, we define a linearization for the aggregate demand $U_k(t)$, in the same way as in Equation (15) for individual demands, with own-price elasticity variables:

$$U_k(t) = U_k^0(t) - \sum_{j=1}^{m} zm_{j,k}(t) + \sum_{j=1}^{n} zn_{j,k}(t) \tag{19}$$

$$U_k^0(t) = \sum_{i \in I(k)}^{m} DM_i^0(t)$$

The component demands are linearized in proportion to the aggregate demand, using shares according to the baseline demands, and then adding the substitution elasticity variables:

$$DM_i(t) = \alpha_{ki}(t) \cdot U_k(t) - \sum_{j=1}^{m} sm_{j,i}(t) + \sum_{j=1}^{n} sn_{j,i}(t) \quad \forall i \in I(k) \tag{20}$$

$$\alpha_{ki}(t) = \frac{DM_i^0(t)}{U_k^0(t)}, \ \sum_{i \in I(k)} \alpha_{ki}(t) = 1$$

Hence, we can also write $U_k(t)$ in terms of the component demands:

$$U_k(t) = \sum_{i \in I(k)} \left(DM_i(t) + \sum_{j=1}^{m} sm_{j,i}(t) - \sum_{j=1}^{n} sn_{j,i}(t) \right) \tag{21}$$

This relation shows that the aggregate demand equals the sum of the component demands *before* substitution. Finally, we need to ensure that aggregate demand is still satisfied *after* substitution, taking into account the utility variation due to the price changes of the component demands. This can be done by imposing the following balance:

$$\begin{aligned} U_k(t) = \sum_{i \in I(k)} \left(\frac{p_i^0(t)}{p_{U_k}^0(t)} \left(DM_i(t) + \sum_{j=1}^{m} sm_{j,i}(t) - \sum_{j=1}^{n} sn_{j,i}(t) \right) - \sum_{j=1}^{m} \frac{p_{ji}^-(t)}{p_{U_k}^0(t)} \cdot sm_{j,i}(t) + \sum_{j=1}^{n} \frac{p_{ji}^+(t)}{p_{U_k}^0(t)} \cdot sn_{j,i}(t) \right) = \\ \sum_{i \in I(k)} \left(\frac{p_i^0(t)}{p_{U_k}^0(t)} DM_i(t) + \sum_{j=1}^{m} \frac{(p_i^0(t) - p_{ji}^-(t))}{p_{U_k}^0(t)} \cdot sm_{j,i}(t) - \sum_{j=1}^{n} \frac{(p_i^0(t) - p_{ji}^+(t))}{p_{U_k}^0(t)} \cdot sn_{j,i}(t) \right) \end{aligned} \tag{22}$$

With respect to the objective function, the price of each demand should be equal to the marginal utility in the optimal solution. As the own price elasticity of the CES component demands are equal with the common substitution elasticity ($E_i = -\sigma_k$, for $i \in I_k$), we can use the same objective terms $dc_i(t) \times DM_i(t)^{1+1/E}$ as those presented in Section 3.1, even though Equation (22) in fact already guarantees that the sum of utility variations is zero over each group of CES component demands in the optimal solution. In the linearized case, the component demands DM_{ki} of each aggregate demand U_k are thus not excluded from the set of independently handled demands DM_i in the objective function (12).

For the aggregate demand $U_k(t)$, we then only need to include the variation in the aggregate utility, $\Delta[U_k(t)^{1+1/E}]$, due to its own-price elasticity. Consequently, we can write the linearized terms in the objective function corresponding to Equation (12) as follows:

$$\begin{aligned} DM_i(t)^{1+1/E_i} &\cong DM_i^0(t)^{1+1/E_i} - \sum_{j=1}^{m} AM_{j,i}(t) \bullet sm_{j,i}(t) + \sum_{j=1}^{n} AN_{j,i}(t) \bullet sn_{j,i}(t) \\ \Delta\left[U_k(t)^{1+1/E_k}\right] &\cong -\sum_{j=1}^{m} BM_{j,k}(t) \bullet zm_{j,k}(t) + \sum_{j=1}^{n} BN_{j,k}(t) \bullet zn_{j,k}(t) \end{aligned} \qquad (23)$$

The coefficients $AM_{j,i}$(t) and $AN_{j,i}$(t) are defined as shown earlier, and the coefficients $BM_{j,k}$ and $BN_{j,k}$ are defined in the same fashion for the aggregate demand $U_k(t)$. In the results, the level of $U_k(t)$ represents the value of the CES utility function (5), and its marginal represents the aggregate price (7).

Note that in the formulation presented above, we have also assumed that the bounds for the step variables $sm_{j,i}(t)$ and $sn_{j,i}(t)$ are now defined in proportion to the ratio of the aggregate demand $U_k(t)/U_k^0(t)$ instead of absolute bounds.

3.3 Customized substitution rates

The linearization presented in Section 3.2 assumes that the price ratios determine the marginal substitution rates, as in the non-linear CES formulation. However, in some cases one might want to assume different substitution rates that better correspond to the physical substitution of the demands in question.

The linearized formulation implemented in TIMES allows two different variants for supporting such custom substitution rates $dag_{ki}(t)$, for which the user should provide exogenously estimated values:

1. The user-defined substitution rates $dag_{ki}(t)$ will be normalized, such that $U_k^0(t) = \sum_i dag_{ki}(t) \cdot DM_i^0(t) = \sum_i DM_i^0(t)$, and are then applied as multipliers to the $DM_i(t)$ variables in Equation (22), instead of the standard price ratios $p_i^0 / p_{U_k}^0$. The resulting demand substitution will then be primarily based on the user-defined rates, but will still take into account the changes in the price ratios.

2. Volume preserving substitution, through relaxing (22) but additionally requiring that insofar as the aggregate demand remains constant, it will remain equal to the weighted sum of the component demands:

$$U_k(t) = \sum_i dag_{ki}(t) \cdot DM_i(t) - \sum_{j=1}^{m} sm_{j,i}(t) + \sum_{j=1}^{n} sn_{j,i}(t) \qquad (24)$$

However, one should note that unlike the implementation of the standard CES function, neither of these two variants preserve the standard property of substitution elasticities, which for a pair of two commodities can be written as:

$$\sigma_{i,j} = -\frac{\partial\left(\frac{DEM_i}{DEM_j}\right)\left(\frac{p_i}{p_j}\right)}{\partial\left(\frac{p_i}{p_j}\right)\left(\frac{DEM_i}{DEM_j}\right)} \tag{25}$$

There are also variants of the CES function, which do preserve both the volume and the substitution elasticity property (25) (Mensbrugghe & Peters 2016). For now, such a variant has not been implemented in TIMES, due to the additional complexities involved in linearizing such a special volume-preserving CES.

4 TIMES User Interface for Demand Functions

4.1 Input parameters

The input parameters for defining the TIMES demand functions are the following:

- COM_PROJ(r,t,c):
  Defines the demand projection for commodity *c* in region *r* and period *t*
- COM_ELAST(r,t,c,s,lim), where lim=LO/UP/FX/N:
  Defines elasticities for demand *c* in region *r* and period *t*, timeslice *s*
  - *lim* = LO/UP :
    defines the own-price elasticity in the lower / upper direction in the linear formulation
  - *lim* = FX (s=ANNUAL) :
    defines own-price elasticities in the non-linear formulation;
    can also be used for defining the own-price elasticities for the aggregate demands in the linear formulation
  - *lim* = N (s=ANNUAL) :
    defines the substitution elasticity for component demands of the demand aggregation represented by commodity *c*; positive values signify standard variant, negative values volume preserving variant
- COM_VOC(r,t,c,bd), where bd=LO/UP:
  Defines the maximum demand variation in the lower / upper direction for demand *c* in region *r* and period *t*
- COM_STEP(r,c,bd), where bd=LO/UP:
  Defines the number of linearization steps in the lower / upper direction for demand *c* in region *r*
- COM_AGG(r,t,c,com):
  Defines an aggregation of component demand *c* into an aggregate demand *com* in region *r* and period *t*; if defined zero (e.g. by IE=2), the values will be auto-generated according to the price ratios.

Important remarks:

- COM_PROJ should be explicitly defined by the user only for the component demands, and never for the aggregate demands.
- As mentioned above, the substitution elasticities can be defined by specifying COM_ELAST(r,t,com,ANNUAL,'N') for the aggregate demands. However, 'FX' elasticities for the *component demands* can be optionally specified for defining component-differentiated substitution elasticities. Nonetheless COM_ELAST(r,t,com,ANNUAL,'N') always defines the minimum substitution elasticity among the component demands of *com*.
- Note that the aggregate demands are always at the ANNUAL level only, and thus only ANNUAL level own-price demand elasticities are supported for the demand aggregates.
- Note that when using the non-linear formulation, only ANNUAL level substitution elasticities are supported also for the component demands of the CES aggregates. The demand variations will thus be proportionally the same for all timeslices.
- Multi-level nested CES demand aggregations are also fully supported both in the non-linear and in the linearized case.
- Recursive CES demand aggregations are not supported, neither in the non-linear nor in the linearized case.
- The Cobb-Douglas case (σ_k=1) is currently handled by setting σ_k very close to unity in the non-linear formulation.

4.2 Usage Notes

Using elastic demands in TIMES always requires that Base prices for the demands are available from a Baseline scenario. The Baseline scenario should thus be run with the following switch:

```
$SET TIMESED NO
```

Apart from the COM_PROJ attribute, the input parameters for elastic demands are only needed in the subsequent policy runs. The elastic demands should be activated in the policy runs with either one of the two following switches (the MICRO switch should be used when using the non-linear formulation):

```
$SET TIMESED YES
$SET MICRO YES
```

The model generator currently automatically prohibits any demand substitution elasticities in the Base year, because the Base year is typically a calibrated historical year, for which the demand prices are also often not well-behaving.

The aggregate demands must be defined to be of type DEM, and the aggregations are defined by specifying the corresponding COM_AGG parameters, as described above. In other respects, the aggregate demands should be dummy commodities, i.e. not included anywhere else in the model RES topology. TIMES may automatically remove from the process topology DEM commodities that can otherwise be identified as aggregate demands, or may discard them as such.

When the substitution elasticities should be different among a group of more than two component demands, there are two approaches available to the user:

1. The user can define a single aggregation group for all of the component demands, and define differentiated elasticities for the individual component demands *cd* by specifying COM_ELAST(r,t,cd,'ANNUAL','FX') in addition to the substitution elasticity σ_0 for the group (see Figure 1). The resulting demand function will not be a proper CES function.
2. The user can define a nested CES structure for the aggregation of the component demands, and define differentiated substitution elasticities for each intermediate aggregation (see Figure 2). The resulting demand function will be a proper nested CES function when positive values σ_i are defined for all of the aggregations and COM_AGG is auto-generated, but the customized linearized substitution variants can also be used.

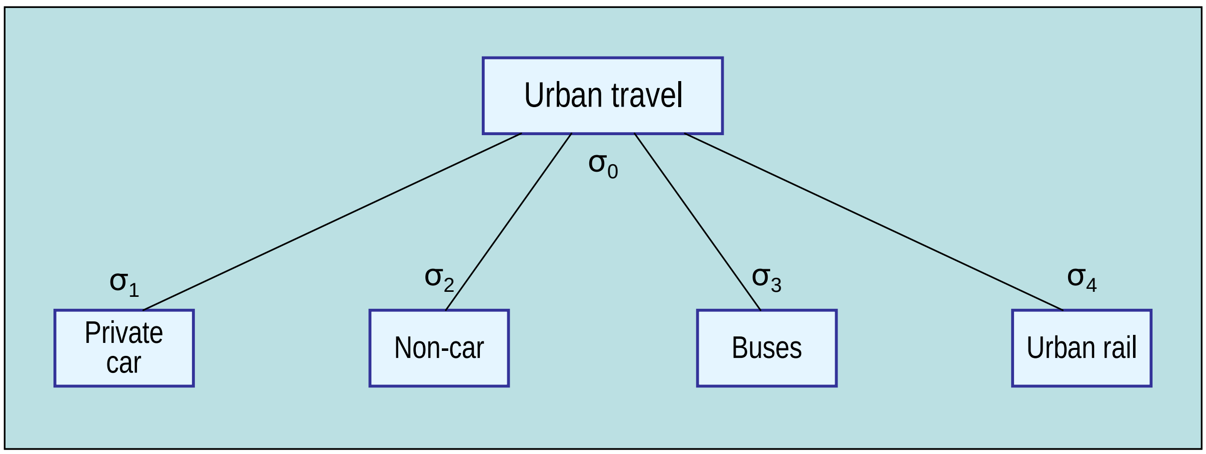

Figure 1. Simple demand aggregation structure for urban transport demand.

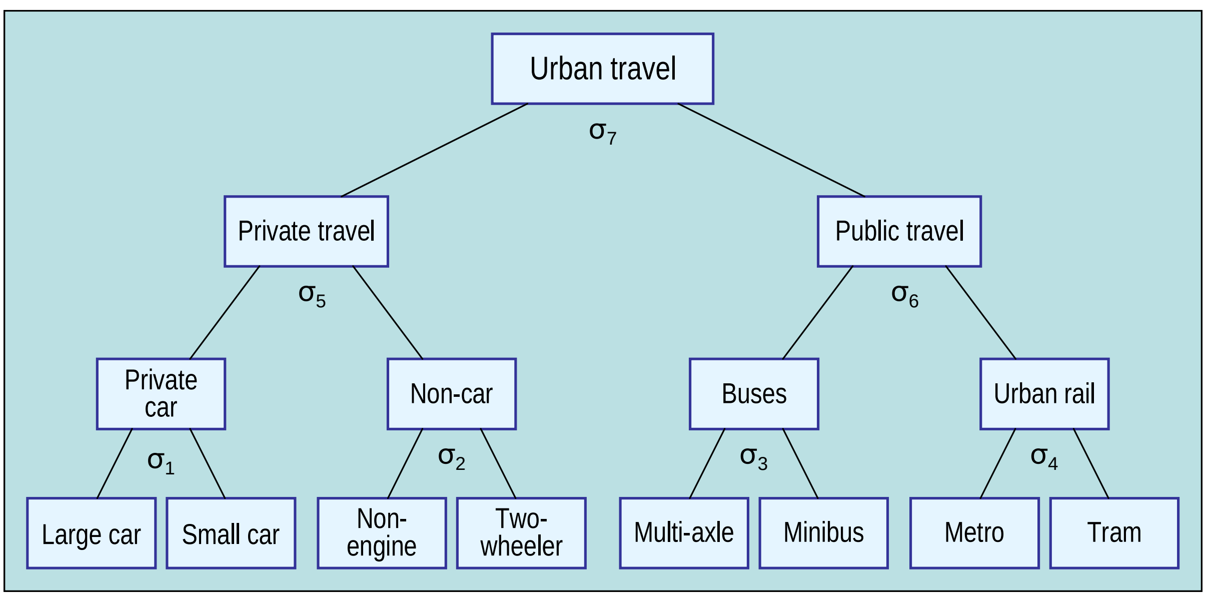

Figure 2. Example nested CES structure for an urban transport demand function.

5 Discussion

This document describes the original TIMES demand functions and a draft design for generalizing the demand function formulations. The generalized design aims to support basic non-linear formulations with demand aggregations defined via CES utility functions. It also implements a linearized formulation for demand aggregates that can be used for modeling cross-price elasticities among the component demands, using either the standard CES substitution rates or customized rates between the demands.

CES functions are being increasingly implemented in economic models integrating engineering and bio-physical properties. The drawback of their use is that they do not preserve additivity, i.e. the sum of the volume components do not add up to the total volume. For example, in the case of land-use, the sum of hectares devoted to different crops by a CES function do not necessarily add up to the total crop-land, and in the case of transportation, the demands allocated to different modes may not reasonably add up to an aggregate demand for passenger or goods transport.

Nonetheless, according to the MARKAL reference manual, CES utility functions could be used e.g. for the aggregate freight and passenger transport. For example, when used for freight transport the relationship would enable some substitution between road and rail transport. However, the substitution rates in the CES aggregates are determined by the price ratios, and so if road and rail transport have different prices, any substitution would lead to a change in the total amount of goods transported, even if the aggregate utility would remain constant. For models that focus on economic aspects, such as the TREMOVE transport model, the consistency of physical transport volumes may be of less importance than for TIMES, and nested CES utility functions are thus often used.

Because in partial equilibrium models the costs of road and rail transport may typically be accounted with different levels of completeness (e.g. infrastructure costs may be quite unaccounted for rail transport), substitution rates driven primarily by price ratios may cause a notable bias between the economic and physical substitution rates. However, the linear formulation supports also user-defined substitution rates in addition to those determined by the price ratios, which might to some extent facilitate more realistic modeling of substitution among e.g. modes of transport. By setting the aggregation rate COM_AGG()=1, one could force 1 tonne-kilometer of rail transport demand to be substituted for each 1 tonne-kilometer of road transport demand, thus retaining the physical volume, unless the total demand decreases due to its own-price elasticity.

All the new generalized formulations have been preliminarily tested and appear to work as expected. Concerning the CES aggregates, the resulting demand variations in the linear formulation have been found equal to those obtained with the non-linear formulation under any combinations of assumed elasticities and price changes for the demands (see one set of example results in Table 1 on next page). The linear formulation thus fully preserves the standard properties of substitution elasticities (25). However, because the granularity caused by the linearization of the demand variations would in practice be typically at least 1%, there will inevitably be some approximation errors due to the coarseness of the linear formulation. The Baseline solution is, of course, also fully replicated when using any of the elastic demand formulations without other model changes.

Finally, one should note that so far the CES formulations have been tested with ANNUAL level demand commodities only, which represents the most commonly used approach in TIMES models. It is quite possible that the linearized CES formulation would need a small additional refinements in order to work correctly with time-sliced demand commodities.

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ANNEX

Table 1. Results for a test model with $\boldsymbol{\sigma_k}$ = 0.9 and E_k = –0.4 for all aggregation groups (Agg-n).

A simple test model was constructed to illustrate the behavior of 9 aggregation groups with different combinations in the component shares and prices. The price of the first component demand in each group is raised by 10%, 20% or 35% from the Baseline level, and the resulting demand levels are then compared between the non-linear and the linearized case. The three bottom rows show the averages over the demand groups. The grid size of the linearized case was chosen to be 0.1%. COM_AGG was auto-generated according to price ratios.

	Baseline case		Non-linear CES formulation			Linear CES formulation			Difference in demand levels		
			+10%	+20%	+35%	+10%	+20%	+35%	+10%	+20%	+35%
Demand	**Level**	**Price**	**Level**	**Level**	**Level**	**Level**	**Level**	**Level**	**Level**	**Level**	**Level**
Dem-1-1	1	1	0.940	0.888	0.823	0.940	0.888	0.823	0.0%	0.0%	0.0%
Dem-1-2	1	1	1.024	1.047	1.079	1.024	1.047	1.079	0.0%	0.0%	0.0%
Agg-1	2	1	1.964	1.935	1.902	1.964	1.935	1.902	0.0%	0.0%	0.0%
Dem-2-1	1	1	0.933	0.875	0.803	0.933	0.875	0.803	0.0%	0.0%	0.0%
Dem-2-2	2	1	2.032	2.062	2.104	2.032	2.061	2.104	0.0%	0.0%	0.0%
Agg-2	3	1	2.965	2.937	2.906	2.965	2.937	2.907	0.0%	0.0%	0.0%
Dem-3-1	2	1	1.895	1.804	1.688	1.895	1.804	1.688	0.0%	0.0%	0.0%
Dem-3-2	1	1	1.032	1.063	1.106	1.032	1.063	1.105	0.0%	0.0%	0.0%
Agg-3	3	1	2.927	2.867	2.794	2.927	2.867	2.793	0.0%	0.0%	0.0%
Dem-4-1	1	1	0.933	0.875	0.803	0.933	0.875	0.803	0.0%	0.0%	0.0%
Dem-4-2	1	2	1.016	1.031	1.052	1.016	1.031	1.052	0.0%	0.0%	0.0%
Agg-4	2	1.5	1.949	1.906	1.855	1.949	1.906	1.855	0.0%	0.0%	0.0%
Dem-5-1	1	2	0.947	0.902	0.844	0.947	0.902	0.844	0.0%	0.0%	0.0%
Dem-5-2	1	1	1.032	1.063	1.106	1.032	1.063	1.105	0.0%	0.0%	0.0%
Agg-5	2	1.5	1.980	1.965	1.950	1.980	1.965	1.949	0.0%	0.0%	0.0%
Dem-6-1	1	1	0.927	0.864	0.787	0.927	0.864	0.787	0.0%	0.0%	0.0%
Dem-6-2	2	2	2.019	2.037	2.062	2.019	2.037	2.062	0.0%	0.0%	0.0%
Agg-6	3	1.67	2.946	2.901	2.849	2.946	2.901	2.848	0.0%	0.0%	0.0%
Dem-7-1	1	2	0.940	0.888	0.823	0.940	0.888	0.823	0.0%	0.0%	0.0%
Dem-7-2	2	1	2.048	2.094	2.157	2.049	2.094	2.157	0.0%	0.0%	0.0%
Agg-7	3	1.33	2.988	2.982	2.980	2.988	2.982	2.980	0.0%	0.0%	0.0%
Dem-8-1	2	1	1.880	1.777	1.646	1.880	1.777	1.646	0.0%	0.0%	0.0%
Dem-8-2	1	2	1.024	1.047	1.079	1.024	1.047	1.079	0.0%	0.0%	0.0%
Agg-8	3	1.33	2.904	2.824	2.725	2.904	2.824	2.725	0.0%	0.0%	0.0%
Dem-9-1	2	2	1.907	1.826	1.722	1.907	1.826	1.722	0.0%	0.0%	0.0%
Dem-9-2	1	1	1.039	1.076	1.128	1.039	1.076	1.128	0.0%	0.0%	0.0%
Agg-9	3	1.67	2.946	2.902	2.850	2.946	2.902	2.850	0.0%	0.0%	0.0%
Aver-1	**1.33**	**1.33**	**1.256**	**1.189**	**1.104**	**1.256**	**1.189**	**1.104**	**0.0%**	**0.0%**	**0.0%**
Aver-2	**1.33**	**1.33**	**1.363**	**1.391**	**1.430**	**1.363**	**1.391**	**1.430**	**0.0%**	**0.0%**	**0.0%**
Agg-Aver	**2.67**	**1.33**	**2.619**	**2.580**	**2.534**	**2.619**	**2.580**	**2.534**	**0.0%**	**0.0%**	**0.0%**